

# DIFFERENCES OF WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS FROM BLOCH-TYPE SPACE TO WEIGHTED-TYPE SPACE

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**ABSTRACT.** We found several new equivalent characterizations for the boundedness of the differences of weighted differentiation composition operators from Bloch-type space to weighted-type space. Especially, we estimated its essential norm in terms of the  $n$ -th power of the induced analytic self-maps on the unit disk, which can provide a new and simple compactness criterion.

## 1. INTRODUCTION AND PRELIMINARIES

Denote  $\mathbb{N}_0$  the set of all nonnegative integers. In the sequel, the notations  $A \approx B$ ,  $A \preceq B$ ,  $A \succeq B$  mean that there maybe different positive constants  $C$  such that  $B/C \leq A \leq CB$ ,  $A \leq CB$ ,  $CB \leq A$ . Let  $H(\mathbb{D})$  be the space of all holomorphic functions on  $\mathbb{D}$  and  $S(\mathbb{D})$  the collection of all holomorphic self-maps on  $\mathbb{D}$ , where  $\mathbb{D}$  is the unit disk in the complex plane  $\mathbb{C}$ . Given a continuous linear operator  $T$  on a Banach space  $X$ , its essential norm is the distance from the operator  $T$  to compact operators on  $X$ , that is,  $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$ . It's trivial that  $\|T\|_e = 0$  if and only if  $T$  is compact, see, e.g. [4] and their references therein.

For  $a \in \mathbb{D}$ , let  $\varphi_a$  be the automorphism of  $\mathbb{D}$  exchanging 0 for  $a$ , that is,  $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ . For  $z, w \in \mathbb{D}$ , the pseudo-hyperbolic distance between  $z$  and  $w$  is given by

$$\rho(z, w) = |\varphi_w(z)| = \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

Immediately, given  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ , we denote  $\rho(z) = \rho(\varphi_1(z), \varphi_2(z))$  for simplicity.

For  $0 < \alpha < \infty$ , an  $f \in H(\mathbb{D})$  is said to be in the Bloch-type space  $\mathcal{B}^\alpha$ , or  $\alpha$ -Bloch space, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

As we all know,  $\mathcal{B}^\alpha$  is a Banach space endowed with the norm  $\|f\|_{\mathcal{B}^\alpha}$ , and the little Bloch-type space  $\mathcal{B}_0^\alpha$  is the closure of polynomials in  $\mathcal{B}^\alpha$ , see, e.g. [3, 5, 10, 11, 16]. In particular,  $\mathcal{B}^\alpha = \mathcal{B}$ , the classical Bloch space for  $\alpha = 1$ ; if  $0 < \alpha < 1$ ,  $\mathcal{B}^\alpha = Lip_{1-\alpha}$ , the analytic Lipschitz space which consists of all  $f \in H(\mathbb{D})$  satisfying

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha},$$

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for some constant  $C > 0$  and all  $z, w \in \mathbb{D}$ ; when  $\alpha > 1$ ,  $\mathcal{B}^\alpha = H_{\alpha-1}^\infty$ , the  $\alpha - 1$  weighted-type space of analytic functions that contains all  $f \in H(\mathbb{D})$  satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1} |f(z)| < \infty.$$

More generally, let  $v$  be a strictly positive continuous and bounded function (weight) on  $\mathbb{D}$ . The weighted-type space  $H_v^\infty$  is defined to be the collection of all functions  $f \in H(\mathbb{D})$  that satisfy

$$\|f\|_v = \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty,$$

provided we identify that differ by a constant, and then  $H_v^\infty$  is a Banach space under the norm  $\|\cdot\|_v$ , see, e.g. [2, 6] and the references therein.

Given  $\varphi \in S(\mathbb{D})$  and  $u \in H(\mathbb{D})$ , the weighted composition operator  $uC_\varphi$  is defined by

$$uC_\varphi(f) = u \cdot (f \circ \varphi) \text{ for } f \in H(\mathbb{D}).$$

As for  $u \equiv 1$ , the weighted composition operator is the usual composition operator, denote by  $C_\varphi$ , see [1]. When  $\varphi = id$  the identity map, the operator  $uC_{id}$  is called multiplication operator  $M_u$ . Let  $D = D^1$  be the differentiation operator, i.e.,  $Df = f'$  for  $f \in H(\mathbb{D})$ . More generally, given an integer  $m \in \mathbb{N}_0$ , we can further define the operator  $D^m f = f^{(m)}$  for  $f \in H(\mathbb{D})$ . Now, the weighted differentiation composition operator, denoted by  $D_{\varphi,u}^m$ , is given as

$$(D_{\varphi,u}^m f)(z) = u(z) f^{(m)}(\varphi(z)), \text{ for } f \in H(\mathbb{D}).$$

In fact, the operator  $D_{\varphi,u}^m$  can degenerate to many classical operators, such as  $D_{\varphi,id}^0 = C_\varphi$  with  $u = id$  and  $m = 0$ ;  $D_{\varphi,u}^0 = uC_\varphi$  with  $m = 0$ ;  $D_{\varphi,id}^1 = C_\varphi D$  with  $u = id$  and  $m = 1$ , and  $D_{\varphi,u}^1 = uC_\varphi D$  with  $m = 1$ .

In 2009, interest has arisen to characterize the properties of composition operator  $C_\varphi$  on Bloch-type spaces in terms of the  $n$ -th power of the analytic self-map  $\varphi$  of the open unit disk  $\mathbb{D}$ . More clearly, Wulan, Zheng and Zhu [14] obtained a new result about the compactness of the composition operator on the Bloch space. It's said that  $C_\varphi$  is compact on the Bloch space  $\mathcal{B}$  if and only if  $\lim_{n \rightarrow \infty} \|\varphi^n\|_{\mathcal{B}} = 0$ , where  $\varphi^n$  means the  $n$ -th power of  $\varphi$ . As regards to Bloch-type spaces, Zhao [15] obtained that  $\|C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta} \approx \limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi^n\|_\beta$  for  $0 < \alpha, \beta < \infty$ . As far as we know that the composition operator is a typical bounded operator on the classical Bloch space  $\mathcal{B}$ , while the differentiation operators are typically unbounded on many Banach spaces of holomorphic functions. Especially, giving the new equivalent characterizations for the boundedness and compactness of weighted differentiation composition operator  $D_{\varphi,u}^m$  are interesting thing, which can unify many classical operators as above. There has been some work on composition and differentiation operators between holomorphic spaces, and the interested readers can refer to [7, 8, 9, 12, 13] and their references therein on much of the developments in the theory of new characterizations. As far as we know, there has been no new similar descriptions for differences of operators. Hence the characterizations for differences of classical operators by the  $n$ -th power of the induced analytic self-maps are in desired need of response. In this paper, we will

try our best to characterize the boundedness and compactness of the operator  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}_\alpha \rightarrow H_v^\infty$ . The paper is organized as follows: we found several characterizations for the boundedness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  in section 2; and then the compactness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  was considered in section 3; finally, some corollaries were presented in section 4.

## 2. THE BOUNDEDNESS OF $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$

In this section, we will give several equivalent characterizations for the boundedness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$ . For  $a \in \mathbb{D}$ , we define the following two families test functions:

$$f_a(z) = \int_0^z \int_0^{t_m} \cdots \int_0^{t_2} \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t_1)^{2\alpha+m-1}} dt_1 dt_2 \cdots dt_m, \quad (2.1)$$

$$g_a(z) = \int_0^z \int_0^{t_m} \cdots \int_0^{t_2} \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t_1)^{2\alpha+m-1}} \cdot \frac{a - t_1}{1 - \bar{a}t_1} dt_1 dt_2 \cdots dt_m. \quad (2.2)$$

Due to the fact  $f \in \mathcal{B}^\alpha$  if and only if  $\|f\|_{\mathcal{B}^\alpha} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+m-1} |f^{(m)}(z)| < \infty$ .

It's obvious that the following equality holds

$$\begin{aligned} \|g_a\|_{\mathcal{B}^\alpha} &\preceq \|f_a\|_{\mathcal{B}^\alpha} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+m-1} |f_a^{(m)}(z)| \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+m-1} \frac{(1 - |a|^2)^\alpha}{|1 - \bar{a}z|^{2\alpha+m-1}} < \infty. \end{aligned}$$

Moreover, by the direct computations, it yields that

$$f_a^{(m)}(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}} \quad \text{and} \quad g_a^{(m)}(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}} \cdot \frac{a - z}{1 - \bar{a}z}. \quad (2.3)$$

For our further use, we denote two notations

$$\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) = \frac{v(z)u_1(z)}{(1 - |\varphi_1(z)|^2)^{\alpha+m-1}}, \quad \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) = \frac{v(z)u_2(z)}{(1 - |\varphi_2(z)|^2)^{\alpha+m-1}};$$

In order to estimate the differences, we prove an estimate for  $|(1 - |z|^2)^{\alpha+m-1} f^{(m)}(z) - (1 - |w|^2)^{\alpha+m-1} f^{(m)}(w)|$  for  $f \in \mathcal{B}^\alpha$  and  $z, w \in \mathbb{D}$ .

**Lemma 2.1.** *Let  $0 < \alpha < \infty$ . Then for each  $f \in \mathcal{B}^\alpha$ , it holds that*

$$|(1 - |z|^2)^{\alpha+m-1} f^{(m)}(z) - (1 - |w|^2)^{\alpha+m-1} f^{(m)}(w)| \leq C \|f\|_{\mathcal{B}^\alpha} \rho(z, w)$$

for all  $z, w \in \mathbb{D}$ .

*Proof.* For  $f \in \mathcal{B}^\alpha$ , it follows that  $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+m-1} |f^{(m)}(z)| < \infty$ . That is to say  $f^{(m)} \in H_{\alpha+m-1}^\infty$ , and moreover  $\|f^{(m)}\|_{H_{\alpha+m-1}^\infty} \preceq \|f\|_{\mathcal{B}^\alpha}$ . By [2, Lemma 3.2], it yields that

$$\begin{aligned} &|(1 - |z|^2)^{\alpha+m-1} f^{(m)}(z) - (1 - |w|^2)^{\alpha+m-1} f^{(m)}(w)| \\ &\preceq \|f^{(m)}\|_{H_{\alpha+m-1}^\infty} \rho(z, w) \preceq \|f\|_{\mathcal{B}^\alpha} \rho(z, w). \end{aligned}$$

This ends the proof.  $\square$

**Lemma 2.2.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following three inequalities hold,*

$$(i) \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) \leq \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \quad (2.4)$$

$$(ii) \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \leq \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \quad (2.5)$$

$$(iii) \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \leq \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \quad (2.6)$$

That is,

$$\begin{aligned} & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\ & \leq \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \end{aligned}$$

*Proof.* For any  $z \in \mathbb{D}$ , we obtain that

$$\begin{aligned} & \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_{\varphi_1(z)}\|_v \geq v(z) |(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_{\varphi_1(z)}(z)| \\ & = v(z) |u_1(z)f_{\varphi_1(z)}^{(m)}(\varphi_1(z)) - u_2(z)f_{\varphi_1(z)}^{(m)}(\varphi_2(z))| \\ & = v(z) \left| \frac{u_1(z)}{(1 - |\varphi_1(z)|^2)^{\alpha+m-1}} - \frac{u_2(z)(1 - |\varphi_1(z)|^2)^\alpha}{(1 - \overline{\varphi_1(z)}\varphi_2(z))^{2\alpha+m-1}} \right| \\ & \geq \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \frac{(1 - |\varphi_1(z)|^2)^\alpha (1 - |\varphi_2(z)|^2)^{\alpha+m-1}}{(1 - \overline{\varphi_1(z)}\varphi_2(z))^{2\alpha+m-1}} \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right| \\ & \geq \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \frac{(1 - |\varphi_1(z)|^2)^\alpha (1 - |\varphi_2(z)|^2)^{\alpha+m-1}}{|1 - \overline{\varphi_1(z)}\varphi_2(z)|^{2\alpha+m-1}} \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|. \end{aligned} \quad (2.7)$$

Similarly, it turns out that

$$\begin{aligned} & \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_{\varphi_1(z)}\|_v \\ & \geq v(z) |u_1(z)g_{\varphi_1(z)}^{(m)}(\varphi_1(z)) - u_2(z)g_{\varphi_1(z)}^{(m)}(\varphi_2(z))| \\ & = v(z) |u_2(z)| \frac{(1 - |\varphi_1(z)|^2)^\alpha}{|1 - \overline{\varphi_1(z)}\varphi_2(z)|^{2\alpha+m-1}} \rho(z) \\ & = \frac{(1 - |\varphi_1(z)|^2)^\alpha (1 - |\varphi_2(z)|^2)^{\alpha+m-1}}{|1 - \overline{\varphi_1(z)}\varphi_2(z)|^{2\alpha+m-1}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z). \end{aligned}$$

On the one hand, we employ the above two inequalities to obtain that

$$\begin{aligned} & |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)|\rho(z) \leq \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_1(z)}\|_v \rho(z) \\ & + \frac{(1 - |\varphi_1(z)|^2)^\alpha (1 - |\varphi_2(z)|^2)^{\alpha+m-1}}{|1 - \overline{\varphi_1(z)}\varphi_2(z)|^{2\alpha+m-1}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|\rho(z) \\ & \leq \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_1(z)}\|_v + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_{\varphi_1(z)}\|_v, \end{aligned} \quad (2.8)$$

where the last inequality follows from  $\rho(z) \leq 1$ . Analogously, we deduce that

$$\begin{aligned} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|\rho(z) & \leq \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_2(z)}\|_v \\ & + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_{\varphi_2(z)}\|_v. \end{aligned} \quad (2.9)$$

From (2.8) and (2.9), we arrive at

$$\begin{aligned} (i) \quad & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)|\rho(z) \\ & \leq \sup_{z \in \mathbb{D}} (\|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_1(z)}\|_v + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_{\varphi_1(z)}\|_v) \\ & \leq \sup_{a \in \mathbb{D}} (\|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_a\|_v + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_a\|_v). \end{aligned} \quad (2.10)$$

$$\begin{aligned} (ii) \quad & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|\rho(z) \\ & \leq \sup_{a \in \mathbb{D}} (\|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_a\|_v + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_a\|_v). \end{aligned} \quad (2.11)$$

On the other hand, we change (2.7) into

$$\begin{aligned} & \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_1(z)}\|_v \\ & = v(z) \left| \frac{u_1(z)}{(1 - |\varphi_1(z)|^2)^{\alpha+m-1}} - \frac{u_2(z)(1 - |\varphi_1(z)|^2)^\alpha}{(1 - \overline{\varphi_1(z)}\varphi_2(z))^{2\alpha+m-1}} \right| \\ & \geq |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| - \frac{v(z)u_2(z)}{(1 - |\varphi_2(z)|^2)^{\alpha+m-1}} \\ & \quad \cdot \left| (1 - |\varphi_1(z)|^2)^{\alpha+m-1} f_{\varphi_1(z)}^{(m)}(\varphi_1(z)) - (1 - |\varphi_2(z)|^2)^{\alpha+m-1} f_{\varphi_1(z)}^{(m)}(\varphi_2(z)) \right| \\ & = |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| - |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\ & \quad \cdot \left| (1 - |\varphi_1(z)|^2)^{\alpha+m-1} f_{\varphi_1(z)}^{(m)}(\varphi_1(z)) - (1 - |\varphi_2(z)|^2)^{\alpha+m-1} f_{\varphi_1(z)}^{(m)}(\varphi_2(z)) \right| \\ & \succeq |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| - |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|\rho(z), \end{aligned}$$

the last inequality is due to Lemma 2.1. From the above inequality it follows that

$$\begin{aligned} (iii) \quad & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\ & \preceq \sup_{z \in \mathbb{D}} (\|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_{\varphi_1(z)}\|_v + |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|\rho(z)) \\ & \preceq \sup_{a \in \mathbb{D}} (\|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)f_a\|_v + \|(D_{\varphi_1,u_1}^m - D_{\varphi_2,u_2}^m)g_a\|_v). \end{aligned} \quad (2.12)$$

(2.8), (2.9) together with (2.12) imply the statement is true. This completes the proof.  $\square$

**Lemma 2.3.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose that  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ , then the following statements hold,*

- (i)  $\sup_{z \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \preceq \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v;$
- (ii)  $\sup_{z \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \preceq \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.$

That is to say,  $\sup_{z \in \mathbb{D}} (\|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v) \preceq \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.$

*Proof.* Recall that

$$\frac{1}{(1 - \bar{a}t_1)^{2\alpha+m-1}} = \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} (\bar{a}t_1)^k.$$

Integrating the above display we express  $f_a$  and  $g_a$  into Maclaurin expansion respectively, as following

$$\begin{aligned} f_a(z) &= (1 - |a|^2)^\alpha \int_0^z \int_0^{t_m} \cdots \int_0^{t_2} \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} (\bar{a}t_1)^k dt_1 dt_2 \cdots dt_m \\ &= (1 - |a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)(k + m)!} (\bar{a})^k z^{k+m}, \quad z \in \mathbb{D}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t_1)^{2\alpha+m-1}} \cdot \frac{a - t_1}{1 - \bar{a}t_1} \\ &= (1 - |a|^2)^\alpha \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \left( \frac{a(1 - \bar{a}t_1) + |a|^2 t_1 - t_1}{1 - \bar{a}t_1} \right) \\ &= (1 - |a|^2)^\alpha \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \left( a - (1 - |a|^2) \frac{t_1}{1 - \bar{a}t_1} \right) \\ &= (1 - |a|^2)^\alpha \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \left( a - (1 - |a|^2) \sum_{k=0}^{\infty} \bar{a}^k t_1^{k+1} \right) \\ &= a(1 - |a|^2)^\alpha \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \\ &\quad - (1 - |a|^2)^{\alpha+1} \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \left( \sum_{k=0}^{\infty} \bar{a}^k t_1^{k+1} \right) \\ &= a(1 - |a|^2)^\alpha \left( \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} \bar{a}^k t_1^k \right) \\ &\quad - (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)l!} \right) \bar{a}^{k-1} t_1^k. \end{aligned}$$

$$\begin{aligned}
g_a(z) &= \int_0^z \int_0^{t_m} \cdots \int_0^{t_2} \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t_1)^{2\alpha+m-1}} \cdot \frac{a-t}{1 - \bar{a}t_1} dt_1 \cdots dt_m \\
&= af_a(z) - (1 - |a|^2)^{\alpha+1} \int_0^z \cdots \int_0^{t_2} \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha+m-1)}{\Gamma(2\alpha+m-1)l!} \bar{a}^{k-1} t_1^l \right) dt_1 \cdots dt_m \\
&= af_a(z) - (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} \frac{k!}{(k+m)!} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha+m-1)}{\Gamma(2\alpha+m-1)l!} \bar{a}^{k-1} z^{k+m} \right). \quad (2.13)
\end{aligned}$$

On the other hand, we deduce that

$$\begin{aligned}
&\|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
&\leq (1 - |a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha+m-1)}{\Gamma(2\alpha+m-1)(k+m)!} |\bar{a}|^k \frac{(k+m)!}{k!} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&= (1 - |a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha+m-1)}{\Gamma(2\alpha+m-1)k!} |\bar{a}|^k \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \quad (2.14)
\end{aligned}$$

$$\begin{aligned}
&\leq (1 - |a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha+m-1)}{\Gamma(2\alpha+m-1)k!} |\bar{a}|^k k^{-\alpha-m+1} \\
&\quad \cdot \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v. \quad (2.15)
\end{aligned}$$

By Stirling's formula, it follows that

$$\begin{cases} \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} \approx k^{\alpha-1}, & \text{as } k \rightarrow \infty; \\ \frac{\Gamma(k+2\alpha+m-1)}{\Gamma(2\alpha+m-1)k!} k^{-\alpha-m+1} \approx k^{\alpha-1}, & \text{as } k \rightarrow \infty. \end{cases}$$

Therefore it yields that

$$\begin{aligned}
\frac{1}{(1 - |a|)^\alpha} &= \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} |a|^k \\
&\approx \sum_{k=0}^{\infty} k^{\alpha-1} |a|^k \\
&\approx \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha+m-1)}{\Gamma(2\alpha+m-1)k!} k^{-\alpha-m+1} |\bar{a}|^k. \quad (2.16)
\end{aligned}$$

Putting (2.16) into (2.15), we deduce that

$$\|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \preceq \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \quad (2.17)$$

Using (2.13) it turns out that

$$\begin{aligned}
& \| (D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m) g_a \|_v \preceq \| (D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m) f_a \|_v \\
& + (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} \frac{k!}{(k+m)!} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha+m-1)}{\Gamma(2\alpha+m-1)l!} \right) |\bar{a}|^{k-1} \\
& \cdot \frac{(k+m)!}{k!} \| u_1 \varphi_1^k - u_2 \varphi_2^k \|_v \\
& = \| (D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m) f_a \|_v \\
& + (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} \left( \sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha+m-1)}{\Gamma(2\alpha+m-1)l!} \right) |\bar{a}|^{k-1} \| u_1 \varphi_1^k - u_2 \varphi_2^k \|_v.
\end{aligned}$$

Furthermore by Stirling's formula again, we obtain

$$\begin{aligned}
& \sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha+m-1)}{\Gamma(2\alpha+m-1)l!} \\
& \approx \sum_{l=0}^{k-1} l^{2\alpha+m-2} \\
& \approx k^{2\alpha+m-1}, \text{ as } k \rightarrow \infty.
\end{aligned}$$

For simplicity, we denote

$$a_k = \sum_{l=0}^{k-1} l^{2\alpha+m-2},$$

and then the last equivalence is due to the following fact,

$$\begin{aligned}
& k^{2\alpha+m-1} - (k-1)^{2\alpha+m-1} \\
& = (k-1+1)^{2\alpha+m-1} - (k-1)^{2\alpha+m-1} \\
& = (k-1)^{2\alpha+m-1} + (2\alpha+m-1)(k-1)^{2\alpha+m-2} + \dots + 1 - (k-1)^{2\alpha+m-1}.
\end{aligned}$$

We deduce from Stole formula that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \frac{a_k}{k^{2\alpha+m-1}} \\
& = \lim_{k \rightarrow \infty} \frac{a_k - a_{k-1}}{k^{2\alpha+m-1} - (k-1)^{2\alpha+m-1}} \\
& = \lim_{k \rightarrow \infty} \frac{(k-1)^{2\alpha+m-2}}{k^{2\alpha+m-1} - (k-1)^{2\alpha+m-1}} \\
& = \lim_{k \rightarrow \infty} \frac{(k-1)^{2\alpha+m-2}}{(2\alpha+m-1)(k-1)^{2\alpha+m-2} + \dots + 1} \\
& = \frac{1}{2\alpha+m-1}.
\end{aligned}$$



Therefore, it yields that

$$\begin{aligned}
& \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \preceq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
& + (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
& \leq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
& + (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} k^{2\alpha+m-1} |\bar{a}|^{k-1} k^{-\alpha-m+1} \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \\
& \preceq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
& + (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} k^{\alpha} |\bar{a}|^{k-1} \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \\
& \preceq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
& + (1 - |a|^2)^{\alpha+1} \frac{1}{(1 - |a|)^{\alpha+1}} \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \\
& \preceq \sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.
\end{aligned} \tag{2.18}$$

This completes the proof.  $\square$

In this section , our main result is exhibited below:

**Theorem 2.4.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following statements are equivalent,*

- (i)  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded;
- (ii)

$$\begin{aligned}
& \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| < \infty, \\
& \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| < \infty;
\end{aligned}$$

- (iii)

$$\sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{a \in \mathbb{D}} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v < \infty;$$

- (iv)

$$\sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v < \infty.$$

*Proof.* The implications (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii) yield from Lemma 2.3 and Lemma 2.2, respectively. We only need to prove (i)  $\Rightarrow$  (iv) and (ii)  $\Rightarrow$  (i) below.

(i)  $\Rightarrow$  (iv). Suppose that  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded; that is,  $\|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} < \infty$ . Considering the function  $z^n$ , we have known that

$\|z^n\|_{\mathcal{B}^\alpha} \approx n^{1-\alpha}$  from [8, Section 2(6)]. And then we obtain that

$$\begin{aligned}
& \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} \\
& \preceq \left\| (D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m) \frac{z^n}{\|z^n\|_{\mathcal{B}^\alpha}} \right\|_v \\
& \preceq \frac{n!}{(n-m)!n^{1-\alpha}} \|u_1 \varphi_1^{n-m} - u_2 \varphi_2^{n-m}\|_v \\
& \approx (n-m)^{\alpha+m-1} \|u_1 \varphi_1^{n-m} - u_2 \varphi_2^{n-m}\|_v \\
& \approx n^{\alpha+m-1} \|u_1 \varphi_1^{n-m} - u_2 \varphi_2^{n-m}\|_v.
\end{aligned} \tag{2.19}$$

(2.19) implies

$$\sup_{n \in \mathbb{N}_0} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \preceq \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} < \infty,$$

then the implication (i)  $\Rightarrow$  (iv) follows.

(ii)  $\Rightarrow$  (i). For any  $f \in \mathcal{B}^\alpha$ , we employ Lemma 2.1 to show that

$$\begin{aligned}
& \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f\|_{H_v^\infty} = \sup_{z \in \mathbb{D}} v(z) |u_1(z)f^{(m)}(\varphi_1(z)) - u_2(z)f^{(m)}(z)| \\
& \preceq \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| |(1-|\varphi_1|^2)^{\alpha+m-1}f^{(m)}(\varphi_1(z)) - (1-|\varphi_2|^2)^{\alpha+m-1}f^{(m)}(\varphi_2(z))| \\
& + \sup_{z \in \mathbb{D}} (1-|\varphi_2|^2)^{\alpha+m-1} |f^{(m)}(\varphi_2(z))| |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\
& \preceq \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| < \infty. \tag{2.20}
\end{aligned}$$

Analogously to (2.20), we can also obtain that

$$\begin{aligned}
& \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f\|_{H_v^\infty} \\
& \preceq \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| < \infty. \tag{2.21}
\end{aligned}$$

The above two inequalities imply that each one of conditions (ii) can ensure the boundedness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$ . This finishes the proof.  $\square$

### 3. THE COMPACTNESS OF $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$

In this section, we give some lemmas to provide the results for the compactness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$ . Firstly, we give some parallel results from Lemma 2.2 as follows.

**Lemma 3.1.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following inequalities hold,*

$$\begin{aligned}
& (i) \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) \\
& \preceq \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \\
& (ii) \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\
& \preceq \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v. \\
& (iii) \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\
& \preceq \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v.
\end{aligned}$$

*Proof.* This results can be deduced from (2.8)–(2.12) in Lemma 2.2.  $\square$

**Lemma 3.2.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded, then the following statements hold,*

$$(i) \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \preceq \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1\varphi_1^n - u_2\varphi_2^n\|_v. \quad (3.1)$$

$$(ii) \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \preceq \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1\varphi_1^n - u_2\varphi_2^n\|_v. \quad (3.2)$$

*Proof.* For any  $a \in \mathbb{D}$  and each positive integer  $N$ , employing (2.14) we obtain

$$\begin{aligned}
& \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
& \leq (1 - |a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1\varphi_1^k - u_2\varphi_2^k\|_v \\
& \leq (1 - |a|^2)^\alpha \sum_{k=0}^N \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1\varphi_1^k - u_2\varphi_2^k\|_v \\
& \quad + (1 - |a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1\varphi_1^k - u_2\varphi_2^k\|_v.
\end{aligned}$$

We denote

$$\begin{aligned}
J_1 &:= (1 - |a|^2)^\alpha \sum_{k=0}^N \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1\varphi_1^k - u_2\varphi_2^k\|_v; \\
J_2 &:= (1 - |a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1\varphi_1^k - u_2\varphi_2^k\|_v.
\end{aligned}$$

For  $k \in \{0, \dots, N\}$ , we can choose  $z^{k+m} \in \mathcal{B}^\alpha$ . Using the boundedness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$ , it turns out that  $\|u_1\varphi_1^k - u_2\varphi_2^k\|_v < \infty$ . Hence  $\limsup_{|a| \rightarrow 1} J_1 = 0$ .

On the other hand, it follows from (2.16) that

$$\begin{aligned}
J_2 &= (1 - |a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&\leq (1 - |a|^2)^\alpha \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\alpha + m - 1)}{\Gamma(2\alpha + m - 1)k!} |\bar{a}|^k k^{-\alpha-m+1} \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \\
&\preceq \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.
\end{aligned}$$

Furthermore, letting  $|a| \rightarrow 1$ , it leads to

$$\limsup_{|a| \rightarrow 1} J_2 \preceq \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.$$

The above inequalities imply that (3.1) is true. Similarly, by (2.18)

$$\begin{aligned}
&\|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \preceq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=1}^{\infty} k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&\leq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=1}^N k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=N+1}^{\infty} k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&\leq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=1}^N k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=N+1}^{\infty} k^{2\alpha+m-1} |\bar{a}|^{k-1} k^{-\alpha-m+1} \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \\
&\preceq \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v \\
&+ (1 - |a|^2)^{\alpha+1} \sum_{k=1}^N k^{2\alpha+m-1} |\bar{a}|^{k-1} \|u_1 \varphi_1^k - u_2 \varphi_2^k\|_v \\
&+ \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.
\end{aligned}$$

Letting  $|a| \rightarrow 1$  in the above display, we get that

$$\begin{aligned}
&\limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \\
&\preceq \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \sup_{n \geq N+1} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v.
\end{aligned}$$

The above inequality together with (3.1) verify (3.2). This ends the proof.  $\square$

Here we will use the similar methods in [8, Section 4], we let  $K_r f(z) = f(rz)$  for  $r \in (0, 1)$ . And then  $K_r$  is a compact operator on the Bloch type space  $\mathcal{B}^\alpha$  or the little Bloch type space  $\mathcal{B}_0^\alpha$  for  $\alpha > 0$  with  $\|K_r\| \leq 1$ . Here we combine the cases for  $0 < \alpha < 1$ ,  $\alpha = 1$  and  $\alpha > 1$  for Bloch type space.

**Lemma 3.3.** [15, Lemma 4.1-4.3] *Let  $0 < \alpha < \infty$ . Then there is a sequence  $\{r_k\}$ , with  $0 < r_k < 1$  tending to 1, such that the compact operator*

$$L_n = \frac{1}{n} \sum_{k=1}^n K_{r_k}$$

*on  $\mathcal{B}_0^\alpha$  satisfies  $\lim_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|z| \leq t} |(I - L_n)f'(z)| = 0$  for any  $t \in [0, 1)$ . Furthermore, this statement holds as well for the sequence of biadjoints  $L_n^{**}$  on  $\mathcal{B}^\alpha$ .*

The following is our main theorem in this section.

**Theorem 3.4.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $D_{\varphi_i, u_i}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ , then the following equivalences hold,*

$$\begin{aligned} & \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{e, \mathcal{B}^\alpha \rightarrow H_v^\infty} \\ & \approx \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) \\ & \quad + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & \quad + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\ & \approx \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \\ & \approx \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v. \end{aligned}$$

*Proof.* Firstly, the boundedness of  $D_{\varphi_i, u_i}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  implies that  $M_i = \sup_{z \in \mathbb{D}} v(z)|u_i(z)| < \infty$  for  $i = 1, 2$ . Lemma 2.3 together with Lemma 3.1 ensure that

$$\begin{aligned} & \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & \quad + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \\ & \leq \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v \\ & \leq \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v. \end{aligned}$$

In the following, we only need to show

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v \leq \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{e, \mathcal{B}^\alpha \rightarrow H_v^\infty} \\ & \leq \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & \quad + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|. \end{aligned}$$

The first inequality follows from the fact: choose a sequence  $f_n(z) = z^n / \|z^n\|_{\mathcal{B}^\alpha}$ , which converges to 0 in  $\mathcal{B}^\alpha$  with  $\|f_n\|_{\mathcal{B}^\alpha} = 1$ . For any compact operator  $K : \mathcal{B}^\alpha \rightarrow H_v^\infty$ , it yields that  $\lim_{n \rightarrow \infty} \|K f_n\|_v = 0$ . Furthermore, we deduce that

$$\begin{aligned}
& \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{e, \mathcal{B}^\alpha \rightarrow H_v^\infty} \\
& \geq \limsup_{n \rightarrow \infty} \inf_K \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m - K)f_n\|_v \\
& \geq \limsup_{n \rightarrow \infty} \inf_K (\|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_n\|_v - \|K f_n\|_v) \\
& \geq \limsup_{n \rightarrow \infty} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_n\|_v \\
& \geq \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1 \varphi_1^{n-m} - u_2 \varphi_2^{n-m}\|_v,
\end{aligned}$$

the last inequality follows from (2.19).

Now we turn our attention to the second inequality. Let  $\{L_n\}$  be the sequence of operators given in Lemma 3.3. Since  $L_n^{**}$  is compact on  $\mathcal{B}^\alpha$  and  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m$  is bounded from  $\mathcal{B}^\alpha$  to  $H_v^\infty$ , the operator  $(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)L_n^{**} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is also compact. Therefore, it follows that

$$\begin{aligned}
& \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{e, \mathcal{B}^\alpha \rightarrow H_v^\infty} \\
& \leq \limsup_{n \rightarrow \infty} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m) - (D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)L_n^{**}\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} \\
& = \limsup_{n \rightarrow \infty} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)(I - L_n^{**})\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} \\
& = \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)(I - L_n^{**})f\|_v \\
& = \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{z \in \mathbb{D}} v(z) |u_1(z)[(I - L_n^{**})f]^{(m)}(\varphi_1(z)) - u_2(z)[(I - L_n^{**})f]^{(m)}(\varphi_2(z))|.
\end{aligned}$$

For an arbitrary  $r \in (0, 1)$ , we denote

$$\begin{aligned}
\mathbb{D}_1 &= \{z \in \mathbb{D} : |\varphi_1(z)| \leq r, |\varphi_2(z)| \leq r\}, \quad \mathbb{D}_2 = \{z \in \mathbb{D} : |\varphi_1(z)| \leq r, |\varphi_2(z)| > r\}, \\
\mathbb{D}_3 &= \{z \in \mathbb{D} : |\varphi_1(z)| > r, |\varphi_2(z)| \leq r\}, \quad \mathbb{D}_4 = \{z \in \mathbb{D} : |\varphi_1(z)| > r, |\varphi_2(z)| > r\}; \\
I_i &:= \sup_{z \in \mathbb{D}_i} v(z) |u_1(z)[(I - L_n^{**})f]^{(m)}(\varphi_1(z)) - u_2(z)[(I - L_n^{**})f]^{(m)}(\varphi_2(z))|,
\end{aligned}$$

for  $i = 1, 2, 3, 4$ . Then Cauchy's integral formula and Lemma 3.3 imply that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} I_1 \\
& \leq \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_1(z)| \leq r} (v(z)|u_1(z)|) |[(I - L_n^{**})f]^{(m)}(\varphi_1(z))| \\
& \quad + \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_2(z)| \leq r} (v(z)|u_2(z)|) |[(I - L_n^{**})f]^{(m)}(\varphi_2(z))| \\
& \leq M_1 \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_1(z)| \leq r} |[(I - L_n^{**})f]^{(m)}(\varphi_1(z))| \\
& \quad + M_2 \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_2(z)| \leq r} |[(I - L_n^{**})f]^{(m)}(\varphi_2(z))| = 0. \tag{3.3}
\end{aligned}$$

On the other hand, we formulate that

$$\begin{aligned}
& v(z) \left| u_1(z) [(I - L_n^{**})f]^{(m)}(\varphi_1(z)) - u_2(z) [(I - L_n^{**})f]^{(m)}(\varphi_2(z)) \right| \\
& \preceq \frac{v(z) |u_1(z)|}{(1 - |\varphi_1(z)|^2)^{\alpha+m-1}} |(1 - |\varphi_1(z)|^2)^{\alpha+m-1} [(I - L_n^{**})f]^{(m)}(\varphi_1(z)) \\
& - (1 - |\varphi_2(z)|^2)^{\alpha+m-1} [(I - L_n^{**})f]^{(m)}(\varphi_2(z))| \\
& + (1 - |\varphi_2(z)|^2)^{\alpha+m-1} \left| [(I - L_n^{**})f]^{(m)}(\varphi_2(z)) \right| \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right| \\
& \preceq |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + (1 - |\varphi_2(z)|^2)^{\alpha+m-1} \\
& \cdot \left| [(I - L_n^{**})f]^{(m)}(\varphi_2(z)) \right| \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|. \tag{3.4}
\end{aligned}$$

Analogously, we obtain that

$$\begin{aligned}
& v(z) \left| u_1(z) [(I - L_n^{**})f]^{(m)}(\varphi_1(z)) - u_2(z) [(I - L_n^{**})f]^{(m)}(\varphi_2(z)) \right| \\
& \preceq |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) + (1 - |\varphi_1(z)|^2)^{\alpha+m-1} \\
& \cdot \left| [(I - L_n^{**})f]^{(m)}(\varphi_1(z)) \right| \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|. \tag{3.5}
\end{aligned}$$

Now employ Lemma 3.3 and the boundedness of  $\left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|$  in (3.5) to show that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} I_2 \\
& \leq \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\
& + \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \sup_{|\varphi_1(z)| \leq r} (1 - |\varphi_1(z)|^2)^{\alpha+m-1} \left| [(I - L_n^{**})f]^{(m)}(\varphi_1(z)) \right| \\
& \cdot \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right| \\
& \preceq \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z). \tag{3.6}
\end{aligned}$$

Similarly, employing (3.4) we deduce that

$$\limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} I_3 \preceq \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z). \tag{3.7}$$

Finally, we deduce from (3.4) that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} I_4 \preceq \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) \\
& + \|(I - L_n^{**})f\|_{\mathcal{B}^\alpha} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right| \\
& \preceq \sup_{|\varphi_1(z)| > r} \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) \rho(z) \\
& + \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|. \tag{3.8}
\end{aligned}$$

Consequently, (3.5) entails that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} I_4 \preceq \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\
& + \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} \left| \mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z) \right|. \tag{3.9}
\end{aligned}$$

Combining (3.3), (3.6), (3.7) and (3.8), (3.9), we find that

$$\begin{aligned} & \|D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m\|_{e, \mathcal{B}^\alpha \rightarrow H_v^\infty} \\ & \leq \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)|. \end{aligned}$$

This ends all the proof for the essential norm estimation.  $\square$

At last we give three equivalent characterizations for the compactness of  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$ .

**Theorem 3.5.** *Let  $m \in \mathbb{N}_0$ ,  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$ ,  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $D_{\varphi_i, u_i}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ , then  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is compact if and only if one of the following statements hold,*

$$\begin{aligned} (i) & \lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| \rho(z) \\ & + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha+m-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha+m-1}^{\varphi_2}(vu_2)(z)| = 0; \\ (ii) & \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m)g_a\|_v = 0; \\ (iii) & \limsup_{n \rightarrow \infty} n^{\alpha+m-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v = 0. \end{aligned}$$

#### 4. SOME COROLLARIES

In this section, we listed some corollaries for the boundedness and compactness for the difference of several classical operators, such as,  $C_{\varphi_1} - C_{\varphi_2}$ ,  $u_1 C_{\varphi_1} - u_2 C_{\varphi_2}$ ,  $C_{\varphi_1} D - C_{\varphi_2} D$  and  $u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D$  from  $\mathcal{B}^\alpha$  into  $H_v^\infty$ .

**Case I:** Let  $u_1 = u_2 = id$ , the identity map, and  $m = 0$ , then  $D_{\varphi_1, id}^0 - D_{\varphi_2, id}^0 = C_{\varphi_1} - C_{\varphi_2}$ . And in this case, we still use the notations  $f_a$  and  $g_a$  to stand for the following test functions,

$$\begin{aligned} f_a(z) &= \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}}, \\ g_a(z) &= \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}} \cdot \frac{a - z}{1 - \bar{a}z}. \end{aligned}$$

**Corollary 4.1.** *Let  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following statements are equivalent,*

- (i)  $C_{\varphi_1} - C_{\varphi_2} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded;
- (ii)

$$\begin{aligned} & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(v)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(v)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(v)(z)| < \infty, \\ & \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_2}(v)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(v)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(v)(z)| < \infty; \end{aligned}$$



(iii)

$$\sup_{a \in \mathbb{D}} \|(C_{\varphi_1} - C_{\varphi_2})f_a\|_v + \sup_{a \in \mathbb{D}} \|(C_{\varphi_1} - C_{\varphi_2})g_a\|_v < \infty;$$

(iv)

$$\sup_{n \in \mathbb{N}_0} n^{\alpha-1} \|\varphi_1^n - \varphi_2^n\|_v < \infty.$$

**Corollary 4.2.** *Let  $0 < \alpha < \infty$ ,  $v$  be a weight and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $C_{\varphi_i} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ . Then the following statements are equivalent,*

- (i)  $C_{\varphi_1} - C_{\varphi_2} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is compact;
- (ii)  $\lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha-1}^{\varphi_1}(v)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha-1}^{\varphi_2}(v)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha-1}^{\varphi_1}(v)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(v)(z)| = 0;$
- (iii)  $\limsup_{|a| \rightarrow 1} \|(C_{\varphi_1} - C_{\varphi_2})f_a\|_v + \limsup_{|a| \rightarrow 1} \|(C_{\varphi_1} - C_{\varphi_2})g_a\|_v = 0;$
- (iv)  $\limsup_{n \rightarrow \infty} n^{\alpha-1} \|\varphi_1^n - \varphi_2^n\|_v = 0.$

**Case II:** Let  $m = 0$ , then  $D_{\varphi_1, u_1}^0 - D_{\varphi_2, u_2}^0 = u_1 C_{\varphi_1} - u_2 C_{\varphi_2}$ . And in this case, we still use  $f_a$  and  $g_a$  as below.

$$f_a(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}},$$

$$g_a(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha+m-1}} \cdot \frac{a - z}{1 - \bar{a}z}.$$

**Corollary 4.3.** *Let  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$  and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following statements are equivalent,*

- (i)  $u_1 C_{\varphi_1} - u_2 C_{\varphi_2} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded;
- (ii)

$$\sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(vu_2)(z)| < \infty,$$

$$\sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_2}(vu_2)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_{\alpha-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(vu_2)(z)| < \infty;$$

(iii)

$$\sup_{a \in \mathbb{D}} \|(u_1 C_{\varphi_1} - u_2 C_{\varphi_2})f_a\|_v + \sup_{a \in \mathbb{D}} \|(u_1 C_{\varphi_1} - u_2 C_{\varphi_2})g_a\|_v < \infty;$$

(iv)

$$\sup_{n \in \mathbb{N}_0} n^{\alpha-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v < \infty.$$

**Corollary 4.4.** *Let  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$  and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $u_i C_{\varphi_i} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ , then the*

following statements are equivalent,

- (i)  $u_1 C_{\varphi_1} - u_2 C_{\varphi_2} : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is compact;
- (ii)  $\lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_{\alpha-1}^{\varphi_1}(vu_1)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_{\alpha-1}^{\varphi_2}(vu_2)(z)| \rho(z)$   
 $+ \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_{\alpha-1}^{\varphi_1}(vu_1)(z) - \mathcal{T}_{\alpha-1}^{\varphi_2}(vu_2)(z)| = 0;$
- (iii)  $\limsup_{|a| \rightarrow 1} \|(u_1 C_{\varphi_1} - u_2 C_{\varphi_2})f_a\|_v + \limsup_{|a| \rightarrow 1} \|(u_1 C_{\varphi_1} - u_2 C_{\varphi_2})g_a\|_v = 0;$
- (iv)  $\limsup_{n \rightarrow \infty} n^{\alpha-1} \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v = 0.$

**Case III:** Let  $u_1 = u_2 = id$ , the identity map, and  $m = 1$ , it's trivial that  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m = C_{\varphi_1} D - C_{\varphi_2} D$ . And in this case, we still use  $f_a$  and  $g_a$  as below.

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t)^{2\alpha}} dt,$$

$$g_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha}} \cdot \frac{a - t}{1 - \bar{a}t} dt.$$

**Corollary 4.5.** Let  $0 < \alpha < \infty$ ,  $v$  be a weight and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following statements are equivalent,

- (i)  $C_{\varphi_1} D - C_{\varphi_2} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded;
- (ii)  $\sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(v)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(v)(z) - \mathcal{T}_\alpha^{\varphi_2}(v)(z)| < \infty,$   
 $\sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_2}(v)(z)| \rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(v)(z) - \mathcal{T}_\alpha^{\varphi_2}(v)(z)| < \infty;$
- (iii)  $\sup_{a \in \mathbb{D}} \|(C_{\varphi_1} D - C_{\varphi_2} D)f_a\|_v + \sup_{a \in \mathbb{D}} \|(C_{\varphi_1} D - C_{\varphi_2} D)g_a\|_v < \infty;$
- (iv)  $\sup_{n \in \mathbb{N}_0} n^\alpha \|\varphi_1^n - \varphi_2^n\|_v < \infty.$

**Corollary 4.6.** Let  $0 < \alpha < \infty$ ,  $v$  be a weight and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $C_{\varphi_i} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ , then the following statements are equivalent,

- (i)  $C_{\varphi_1} D - C_{\varphi_2} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is compact;
- (ii)  $\lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_\alpha^{\varphi_1}(v)(z)| \rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_\alpha^{\varphi_2}(v)(z)| \rho(z)$   
 $+ \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_\alpha^{\varphi_1}(v)(z) - \mathcal{T}_\alpha^{\varphi_2}(v)(z)| = 0;$
- (iii)  $\limsup_{|a| \rightarrow 1} \|(C_{\varphi_1} D - C_{\varphi_2} D)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(C_{\varphi_1} D - C_{\varphi_2} D)g_a\|_v = 0;$
- (iv)  $\limsup_{n \rightarrow \infty} n^\alpha \|\varphi_1^n - \varphi_2^n\|_v = 0.$

**Case IV:** Let  $m = 1$ , it's trivial that  $D_{\varphi_1, u_1}^m - D_{\varphi_2, u_2}^m = u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D$ . And in this case, we also use  $f_a$  and  $g_a$  to stand for the following test functions.

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t)^{2\alpha}} dt,$$

$$g_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha}} \cdot \frac{a - t}{1 - \bar{a}t} dt.$$

**Corollary 4.7.** *Let  $0 < \alpha < \infty$  and  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$  and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Then the following statements are equivalent,*

- (i)  $u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded;
- (ii)

$$\sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(vu_1)(z)|\rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(vu_1)(z) - \mathcal{T}_\alpha^{\varphi_2}(vu_2)(z)| < \infty,$$

$$\sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_2}(vu_2)(z)|\rho(z) + \sup_{z \in \mathbb{D}} |\mathcal{T}_\alpha^{\varphi_1}(vu_1)(z) - \mathcal{T}_\alpha^{\varphi_2}(vu_2)(z)| < \infty;$$

- (iii)

$$\sup_{a \in \mathbb{D}} \|(u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D)f_a\|_v + \sup_{a \in \mathbb{D}} \|(u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D)g_a\|_v < \infty;$$

- (iv)

$$\sup_{n \in \mathbb{N}_0} n^\alpha \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v < \infty.$$

**Corollary 4.8.** *Let  $0 < \alpha < \infty$ ,  $v$  be a weight. Suppose  $u_1, u_2 \in H(\mathbb{D})$  and  $\varphi_1, \varphi_2 \in S(\mathbb{D})$ . Suppose that  $u_i C_{\varphi_i} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is bounded for  $i = 1, 2$ , then the following statements are equivalent,*

- (i)  $u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D : \mathcal{B}^\alpha \rightarrow H_v^\infty$  is compact;
- (ii)  $\lim_{r \rightarrow 1} \sup_{|\varphi_1(z)| > r} |\mathcal{T}_\alpha^{\varphi_1}(vu_1)(z)|\rho(z) + \lim_{r \rightarrow 1} \sup_{|\varphi_2(z)| > r} |\mathcal{T}_\alpha^{\varphi_2}(vu_2)(z)|\rho(z) + \lim_{r \rightarrow 1} \sup_{\min\{|\varphi_1(z)|, |\varphi_2(z)|\} > r} |\mathcal{T}_\alpha^{\varphi_1}(vu_1)(z) - \mathcal{T}_\alpha^{\varphi_2}(vu_2)(z)| = 0;$
- (iii)  $\limsup_{|a| \rightarrow 1} \|(u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D)f_a\|_v + \limsup_{|a| \rightarrow 1} \|(u_1 C_{\varphi_1} D - u_2 C_{\varphi_2} D)g_a\|_v = 0;$
- (iv)  $\limsup_{n \rightarrow \infty} n^\alpha \|u_1 \varphi_1^n - u_2 \varphi_2^n\|_v = 0.$

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